Exercise 20

Let $f(x) = x^3$.

(a) Estimate the values of $f'(0)$, $f'\left(\frac{1}{2}\right)$, $f'(1)$, $f'(2)$, and $f'(3)$ by using a graphing device to zoom in on the graph of f.

(b) Use symmetry to deduce the values of $f'\left(-\frac{1}{2}\right)$, $f'(-1)$, $f'(-2)$, and $f'(-3)$.

(c) Use the values from parts (a) and (b) to graph f'.

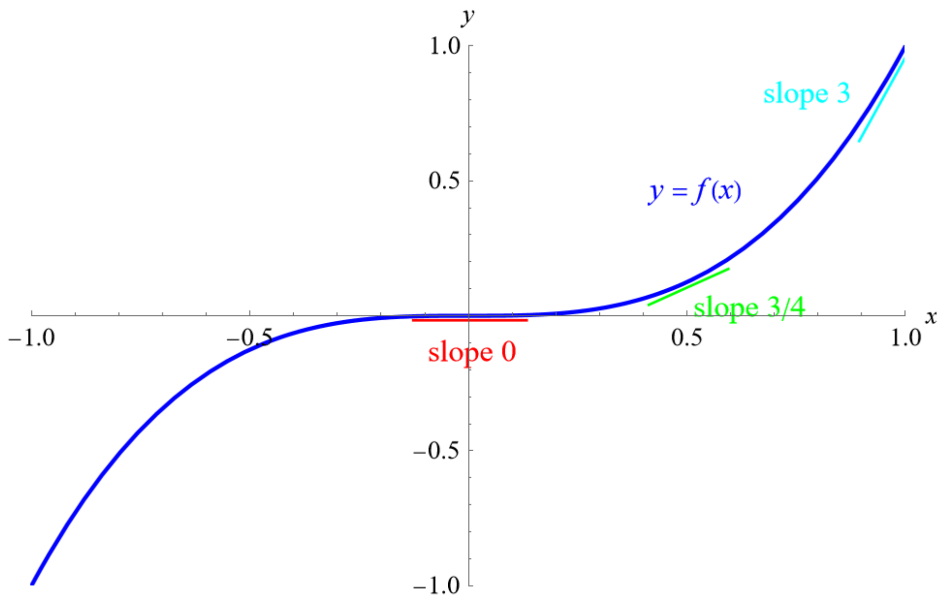
(d) Guess a formula for $f'(x)$.

(e) Use the definition of derivative to prove that your guess in part (d) is correct.

Solution

Part (a)

Draw the tangent lines to the graph at the first few given values of x and estimate their slopes.



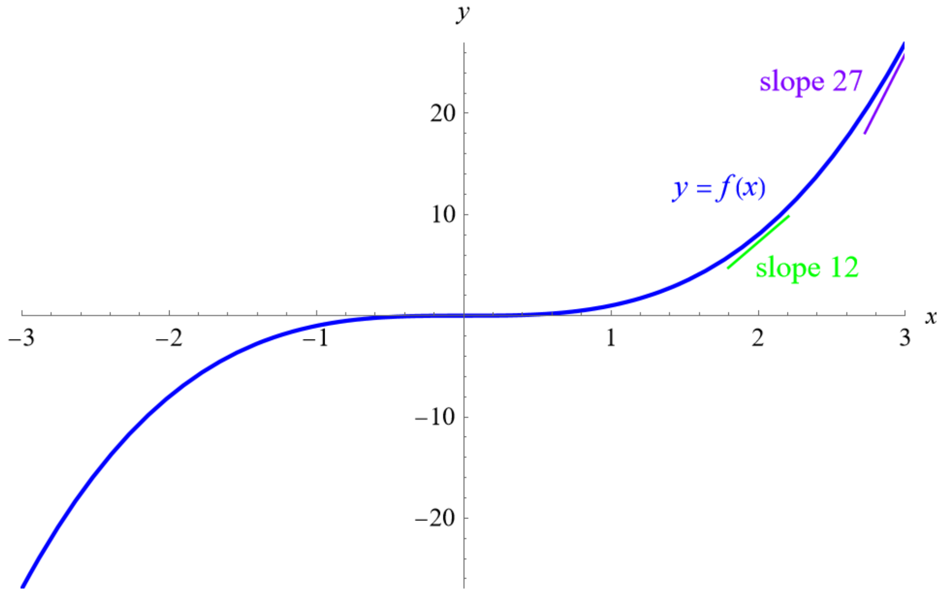
As a result,

$$f'(0) = 0$$

$$f'\left(\frac{1}{2}\right) = \frac{3}{4}$$

$$f'(1) = 3.$$

Draw the tangent lines to the graph at the last few given values of x and estimate their slopes.



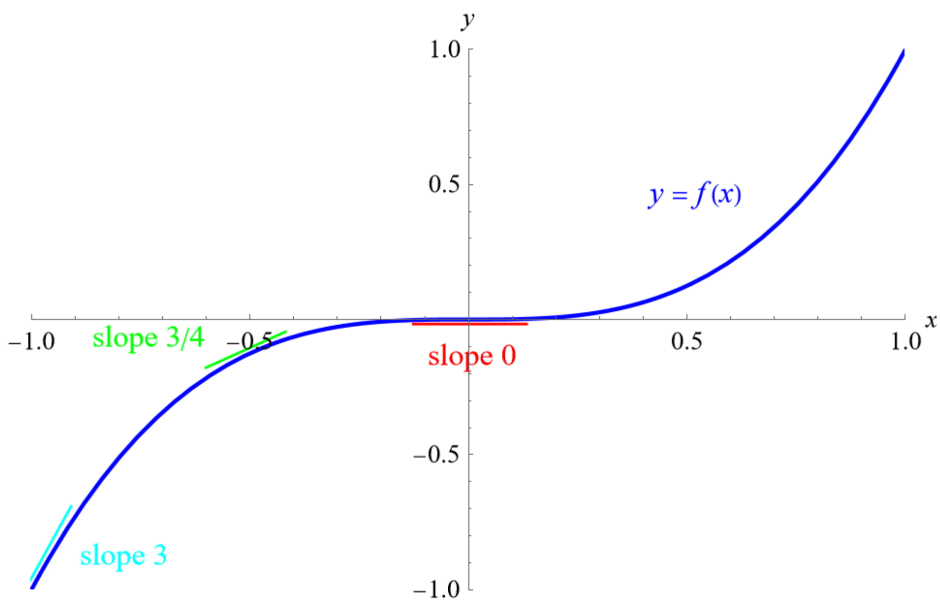
As a result,

$$f'(2) = 12$$

$$f'(3) = 27.$$

Part (b)

Draw the tangent lines to the graph at the first few given values of x and estimate their slopes.

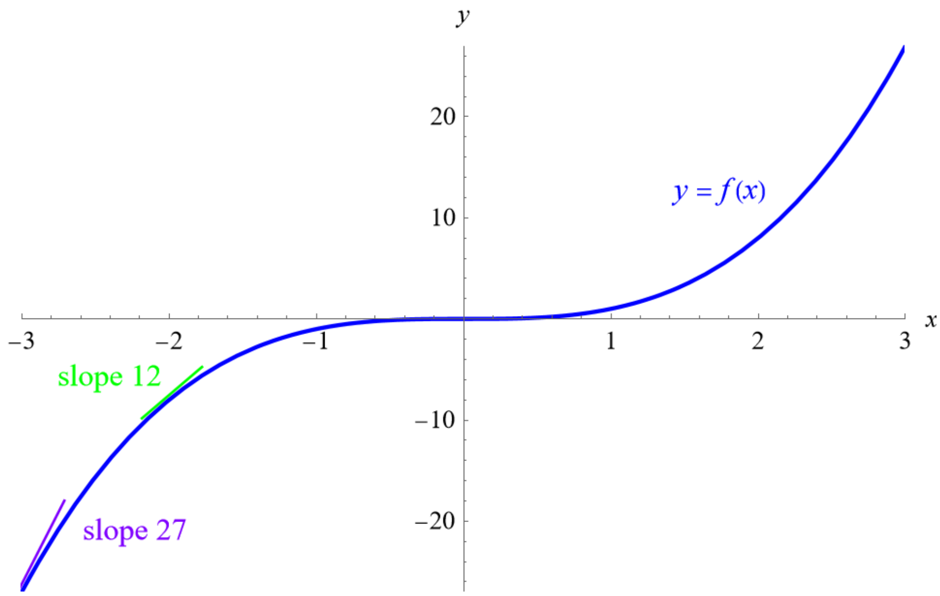


As a result,

$$f'\left(-\frac{1}{2}\right) = \frac{3}{4}$$

$$f'(-1) = 3.$$

Draw the tangent lines to the graph at the last few given values of x and estimate their slopes.



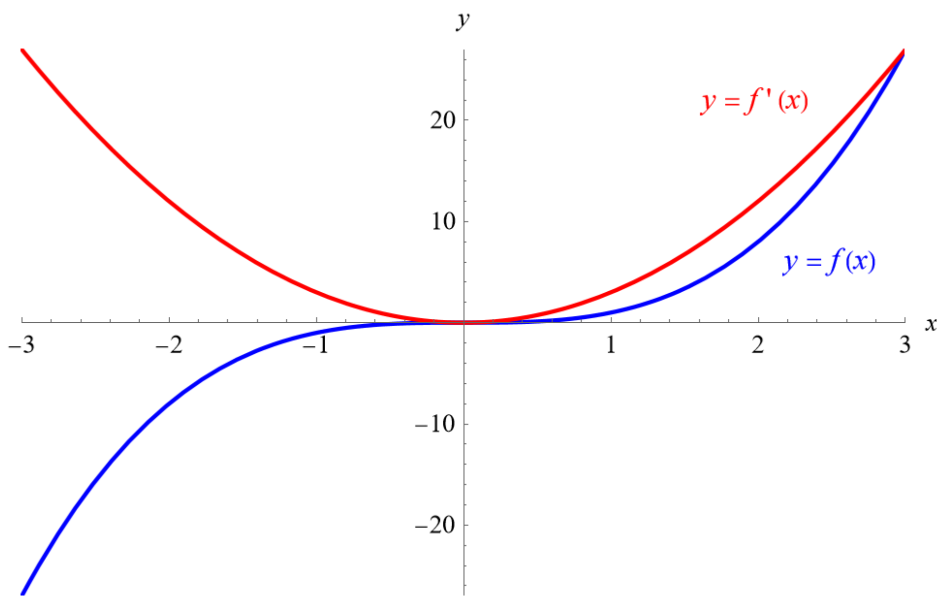
As a result,

$$f'(-2) = 12$$

$$f'(-3) = 27.$$

Part (c)

Below is a graph of $f(x)$ versus x and its derivative.



Part (d)

The derivative of $f(x)$ is

$$f'(x) = 3x^2.$$

Part (e)

Calculate the derivative of $f(x) = x^3$ from the definition.

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\ &= \lim_{h\to 0} \frac{(x+h)^3 - x^3}{h} \\ &= \lim_{h\to 0} \frac{(x^3 + 3x^2h + 3xh^2 + h^3) - x^3}{h} \\ &= \lim_{h\to 0} \frac{3x^2h + 3xh^2 + h^3}{h} \\ &= \lim_{h\to 0} (3x^2 + 3xh + h^2) \\ &= 3x^2 \end{aligned}$$